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Properties of Korovkin Type in $L^p(a, b)$

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INTRODUCTION

Korovkin's well-known theorem asserts that the set of the three functions $\phi_i(t) = t^i$, $i = 0, 1, 2$, is a test set for the convergence to identity of sequences $(L_n)_{n \in \mathbb{N}}$ of positive linear operators on $\mathscr{C}(a, b)$, endowed with the supremum norm. There is no need to make a special hypothesis on the norms of the operators since the property $\lim_{n \to \infty} \|L_n\|_\infty = 1$ is a consequence of the convergence of $L_n\phi_0$ to ϕ_0.

On the contrary, the uniform boundedness of a sequence of positive linear operators L_n is required in order to have $\{\phi_i, i = 0, 1, 2\}$ as a test set for the convergence to identity of L_n on $L^p(a, b)$, $1 \leqslant p < \infty$. It is not a consequence of the convergence for the functions ϕ_i, $i = 0, 1, 2$ (cf. Example 2), and it is necessary according to the uniform boundedness theorem.

The contracting case, on $L^p(a, b)$, was solved by H. Berens and R. A. DeVore [1]. They showed that the functions ϕ_0 and ϕ_1 form a test set for the convergence to identity of positive linear contractions on $L^p(a, b)$ (simultaneously with other authors), and they gave estimates of the order of approximation.

In the first part a Korovkin-type property is studied for the self-adjoint operators on $L^p(a, b)$: these verify, for f and g in $L^p(a, b)$:

$$\int_a^b Lf(x)\, g(x)\, dx = \int_a^b f(x)\, Lg(x)\, dx.$$

The integral operators with symmetrical kernel, convolution operators with positive, even functions, for example, are of this type.

We show that a sequence of self-adjoint positive linear operators converges to identity on $L^p(a, b)$, $p \geqslant 1$, if and only if it converges for ϕ_0 and ϕ_1 and the operators are uniformly bounded. Quantitative results in terms of the first modulus of smoothness and the orders of approximation for the functions ϕ_0 and ϕ_1 are given.

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In the general case, for operators which are not necessary self-adjoint, $\{\phi_0, \phi_1\}$ is no longer a test set. On $L^p(-1, 1)$, let L be defined by

$$Lf(t) = \int_{-1/2}^{1/2} f(x+t)\, dx, \qquad \text{if} \quad |t| \leqslant \tfrac{1}{2}$$

and

$$Lf(t) = f(t), \qquad \text{if} \quad |t| > \tfrac{1}{2}.$$

The constant sequence $L_n = L$ preserves ϕ_0 and ϕ_1 and, of course, does not converge on $L^p(-1, 1)$, $p \geqslant 1$.

In the second part, we give estimates of the order of approximation by a sequence of positive, linear, uniformly bounded, operators L_n in terms of the orders of approximation for ϕ_0, ϕ_1, ϕ_2, and the first modulus of smoothness of f; so we verify that $\{\phi_0, \phi_1, \phi_2\}$ is a test set for the convergence to identity of such positive operators (cf. M. W. Muller [8]).

I. Self-Adjoint Operators

Let $(L_n)_{n \in \mathbb{N}}$ be a sequence of positive linear operators that are self-adjoint. Let

$$\lambda_{n,p,i} = \sup_{j \leqslant i} \| L_n \phi_j - \phi_j \|_{L^p(a,b)}, \qquad i = 0, 1, 2.$$

THEOREM 1. $1 \leqslant p < \infty$. *For every $f \in W^{1,p}(a, b)$ space of absolutely continuous functions such as $f' \in L^p(a, b)$, we have*

$$\| L_n f - f \|_p \leqslant A_p(\lambda_{n,p,0}(\| f \|_1 + \| f' \|_1) + \lambda_{n,1,1}^{\alpha_p} \| f' \|_p)$$

with $\alpha_p = 1/2$ if $p \leqslant 2$ and $\alpha_p = 1/p$ if $p \geqslant 2$.

THEOREM 1'. $1 \leqslant p < \infty$. *If, moreover, the sequence L_n is uniformly bounded on $L^p(a, b)$, for every $f \in L^p(a, b)$, we have*

$$\| L_n f - f \|_p \leqslant B_p(\lambda_{n,p,0} \| f \|_1 + \omega_p(f, \lambda_{n,p,0} + \lambda_{n,1,1}^{\alpha_p})),$$

$\omega_p(f, t)$ being the first modulus of smoothness of f defined by

$$\omega_p(f, t) = \sup_{|u| \leqslant t} \left(\int_{I_u} | f(x+u) - f(x)|^p \, dx \right)^{1/p},$$

where $I_u = \{x \mid x \in (a, b), x + u \in (a, b)\}$, A_p and B_p being constants independent of f and n.

Note. In the above theorems the equality holds for the constants. Examples in part III will be given in order to show that, in some sense, these estimates are the best possible.

If $\| L_n\phi_0 \|_\infty$ is bounded in n, we have $\lambda_{n,p,0} \leqslant \mathrm{Cte}\, \lambda_{n,1,0}^{1/p}$, so the order of approximation in Theorem 1′ is $\omega_p(f, \lambda_{n,1,1}^{\alpha_p})$.

We show first that the general problem can be reduced to this case. We introduce, for $n \in \mathbb{N}$, the set $E_n = \{x \mid x \in (a, b),\ |L_n\phi_0(x) - \phi_0(x)| \geqslant 1\}$ and E_n^c is the complementary set of E_n in (a, b).

LEMMA 1.1. $1 \leqslant p < \infty$. *For every* $f \in W^{1,p}(a, b)$, *we have*

$$\| 1_{E_n}(L_n f - f)\|_p \leqslant 3(\| f' \|_1 + (b-a)^{-1} \| f \|_1)\, \lambda_{n,p,0},$$

where 1_X *is the indicator function of the set* X.

Proof. Let $\mu(E_n)$ be the Lebesgue measure of E_n. We have

$$\mu(E_n) \leqslant \int_{E_n} |L_n\phi_0(x) - \phi_0(x)|^p\, dx \leqslant \| L_n\phi_0 - \phi_0 \|_p^p = \lambda_{n,p,0}^p.$$

On the other hand, for every $f \in W^{1,p}(a, b)$, the inequality $\| f \|_\infty \leqslant (b-a)^{-1} \| f \|_1 + \| f' \|_1$ holds; it is a consequence of $|f(x)| \leqslant |f(t)| + \| f' \|_1$, for every $(x, t) \in (a, b)^2$. Then, we write

$$\begin{aligned} \| 1_{E_n}(L_n f - f)\|_p &\leqslant \| f \|_\infty\, (\| 1_{E_n} L_n\phi_0 \|_p + \| 1_{E_n}\phi_0 \|_p) \\ &\leqslant \| f \|_\infty\, (\| L_n\phi_0 - \phi_0 \|_p + 2\, \| 1_{E_n}\phi_0 \|_p) \\ &\leqslant 3\lambda_{n,p,0}\, \| f \|_\infty. \end{aligned}$$

To study $\| 1_{E_n^c}(L_n f - f)\|_p$, we introduce the function $J(u, x)$ defined on $(a, b)^2$ by

$$J(u, x) = 1 \quad \text{if} \quad u \leqslant x \qquad \text{and} \qquad J(u, x) = 0 \quad \text{if} \quad u > x.$$

We write for every $f \in W^{1,p}(a, b)$ and $(x, t) \in (a, b)^2$:

$$| f(x) - f(t)| = \left| \int_a^b f'(u)(J(u, x) - J(u, t))\, du \right|.$$

Then, we introduce a positive real number h, to be chosen later, in order to part the above interval of integration. We set $L_n^t(f(t), x)$ instead of $L_n f(x)$, to indicate that t is the variable for f and we get results about

$$\| 1_{E_n^c}(x)\, L_n^t(f(x) - f(t), x)\|_p$$

in the following lemmas.

LEMMA 1.2. *If $f \in W^{1,p}(a, b)$, we have*

$$\left\| 1_{E_N^c}(x)\, L_n^t\left(\int_t^x f'(u)\, 1_{\{u \mid |x-u|<h\}}(u)\, du, x\right)\right\|_p \leqslant 4h \, \| f' \|_p.$$

Proof. For every $(x, t) \in (a, b)^2$, we use Hölder inequality to get

$$\left|\int_t^x f'(u)\, 1_{\{u \mid |x-u|<h\}}(u)\, du\right|^p$$
$$\leqslant (2h)^{p-1} \int_a^b |f'(u)|^p \, 1_{\{u \mid |x-u|<h\}}(u)\, du$$

and we take L_n of the integral of the left-hand side, considered as a function of t, at the point x.

As L_n is positive and $L_n \phi_0(x) \leqslant 2$ if $x \in E_n^c$, the quantity to bound is less than

$$2(2h)^{1-1/p}\left(\int_a^b \int_a^b |f'(u)|^p \, 1_{\{u \mid |x-u|<h\}}(u)\, du\, dx\right)^{1/p}.$$

LEMMA 1.3. *If $f \in W^{1,p}(a, b)$, the quantities*

$$(b-a)^{2-p}\left|\int_t^x f'(u)\, du\right|^p \qquad \text{if} \quad p \geqslant 2$$

and

$$h^{2-p}\left|\int_t^x f'(u)\, 1_{\{u \mid |x-u|>h\}}(u)\, du\right|^p \qquad \text{if} \quad p \leqslant 2$$

are bounded by

$$\int_a^b |f'(u)|^p \,(x-t)(J(u, x) - J(u, t))\, du.$$

Proof. With the help of Hölder inequality we get

$$\text{if } p \geqslant 2\text{:} \left|\int_t^x f'(u)\, du\right|^p \leqslant |x-t|(b-a)^{p-2}\left|\int_t^x |f'(u)|^p\, du\right|$$

and

$$\text{if } p \leqslant 2\text{:} \left|\int_t^x f'(u)\, 1_{\{u \mid |x-u|>h\}}(u)\, du\right|^p$$
$$\leqslant \left|\frac{x-t}{h}\right|^{2-p} |x-t|^{p-1}\left|\int_t^x |f'(u)|^p\, du\right|.$$

Then we write

$$\left| \int_t^x |f'(u)|^p \, |x-t| \, du \right| = \int_b^a |f'(u)|^p \, (x-t)(J(u,x) - J(u,t)) \, du.$$

LEMMA 1.4. *If $f \in W^{1,p}(a,b)$, we have*

$$\int_a^b L_n^t \left(\int_a^b |f'(u)|^p \, (x-t)(J(u,x) - J(u,t)) \, du, x \right) dx$$
$$\leqslant 4\lambda_{n,1,1} \, \|f'\|_p^p.$$

Proof. We expand the quantity that is under the integral sign and we take L_n of this function of t. Then we use the linearity and the self adjointness of L_n:

$$\int_a^b L_n^t \left(\int_a^b |f'(u)|^p \, (x-t)(J(u,x) - J(u,t)) \, du, x \right) dx$$
$$= 2 \int_a^b |f'(u)|^p \int_a^b J(u,x) \, L_n^t(x-t, x) \, dx \, du$$
$$\leqslant 2 \, \|f'\|_p^p \int_a^b |L_n^t(x-t, x)| \, dx$$
$$\leqslant 2 \, \|f'\|_p^p \, (\|L_n \phi_0 - \phi_0\|_1 + \|L_n \phi_1 - \phi_1\|_1).$$

LEMMA 1.5. *If $f \in W^{1,p}(a,b)$, we have*

$$\| 1_{E_n^c}(x) \, L_n^t(f(x) - f(t), x) \|_p \leqslant C_p \, \|f'\|_p \, \lambda_{n,1,1}^{\alpha_p},$$

with $\alpha_p = 1/2$ if $p \leqslant 2$ and $\alpha_p = 1/p$ if $p \geqslant 2$.

Proof. If $p \geqslant 2$ and $x \in E_n^c$, we use Hölder inequality for the positive linear operators and Lemma 1.3:

$$|L_n^t(f(x) - f(t), x)|^p \leqslant (b-a)^{p-2} \, (L_n \phi_0(x))^{p-1}$$
$$\times L_n^t \left(\int_a^b |f'(u)|^p \, (x-t)(J(u,x) - J(u,t)) \, du, x \right).$$

Then, with the help of Lemma 1.4, we write

$$\| 1_{E_n^c}(x) \, L_n^t(f(x) - f(t), x) \|_p^p \leqslant 2^{p+1} (b-a)^{p-2} \, \|f'\|_p^p \, \lambda_{n,1,1}.$$

If $p \leqslant 2$, the same argument is used to write

$$\left\| 1_{E_n^c}(x)\, L_n^t \left(\int_t^x f'(u)\, 1_{\{u \mid |x-u| > h\}}(u)\, du,\, x \right) \right\|_p \leqslant 2^{1+1/p} h^{1-2/p} \, \| f' \|_p \, \lambda_{n,1,1}^{1/p}.$$

Then we use Lemma 1.2 to get

$$\| 1_{E_n^c}(x)\, L_n^t(f(x) - f(t), x) \|_p \leqslant 4h \, \| f' \|_p + 2^{1+1/p} h^{1-2/p} \, \| f' \|_p \, \lambda_{n,1,1}^{1/p}$$

and we choose $h = \lambda_{n,1,1}^{1/2}$.

Proof of Theorem 1. For $x \in E_n^c$ we write the equality

$$L_n f(x) - f(x) = L_n^t(f(t) - f(x), x) + f(x)(L_n \phi_0(x) - \phi_0(x))$$

and we get the inequality

$$\| L_n f - f \|_p \leqslant \| 1_{E_n}(L_n f - f) \|_p + \| 1_{E_n^c}(x)\, L_n^t(f(t) - f(x), x) \|_p + \| f \|_\infty \, \| L_n \phi_0 - \phi_0 \|_p.$$

Then Lemmas 1.1 and 1.5 give us:

$$\| L_n f - f \|_p \leqslant 4(\| f' \|_1 + (b-a)^{-1} \| f \|_1)\, \lambda_{n,p,0} + C_p \, \| f' \|_p \, \lambda_{n,1,1}^{\alpha_p}.$$

Proof of Theorem 1′. Now we suppose that the sequence (L_n) is uniformly bounded on $L^p(a, b)$ and we set $M = \sup_n \| L_n \|_p$.

We use the Peetre $\mathscr{K}$-functional defined for $f \in L^p(a, b)$ and $t \in (a, b)$ by

$$\mathscr{K}_p(t, f) = \inf_{g \in W^{1,p}(a,b)} (\| f - g \|_p + t \, \| g' \|_p).$$

It verifies $\mathscr{K}_p(t, f) \leqslant \mathrm{Cte}\, \omega_p(f, t)$ and the constant depends only on (a, b). Let $f \in L^p(a, b)$ and $g \in W^{1,p}(a, b)$. We have

$$\| L_n f - f \|_p \leqslant (M+1) \| f - g \|_p + \| L_n g - g \|_p.$$

We use Theorem 1 to bound $\| L_n g - g \|_p$ and we get

$$\| L_n f - f \|_p \leqslant (M + 1 + A_p \lambda_{n,p,0}(b-a)^{1-1/p}) \| g - f \|_p + A_p \lambda_{n,p,0} \, \| f \|_1 + ((b-a)^{1-1/p} A_p \lambda_{n,p,0} + A_p \lambda_{n,1,1}^{\alpha_p}) \| g' \|_p.$$

This inequality is true for every $g \in W^{1,p}(a, b)$, so we have

$$\| L_n f - f \|_p \leqslant \mathrm{Cte}(\lambda_{n,p,0} \, \| f \|_1 + \mathscr{K}_p(\lambda_{n,p,0} + \lambda_{n,1,1}^{\alpha_p}, f)).$$

Remark. H. Berens and R. A. DeVore in [1] gave the following estimate for a sequence (L_n) of positive linear constractions and $f \in L^p(a, b)$,

$$\| L_n f - f \|_p \leqslant C(\lambda_{n,p,1}^{1/p} \| f \|_p + \omega_{2,p}(f, \lambda_{n,p,1}^{1/2p})),$$

where $\omega_{2,p}(f)$ is the second modulus of smoothness.

The moduli of smoothness of orders one and two are related: $\omega_{1,p}(f, t^2) \leqslant \mathrm{Cte}\, \omega_{2,p}(f, t)$ (cf. H. Johnen [5] or G. G. Lorentz [6]).

And for some functions as $\log t$, t^α $(-1 < \alpha < 0)$, we have

$$\omega_{1,p}(f, t) \underset{t \to 0}{\sim} \omega_{2,p}(f, t).$$

Hence, for sequences of self-adjoint positive linear contractions, the estimate of Theorem 1′ is better for $p \geqslant 2$. Moreover, $\lambda_{n,p,1}$ is, in general, of the same order as $\lambda_{n,1,1}^{1/p}$ (Example 5).

II. General Positive Linear Operators

Let $(L_n)_{n \in \mathbb{N}}$ be a sequence of positive linear operators on $L^p(a, b)$, $1 \leqslant p < \infty$.

Theorem 2. *For every $f \in W^{1,p}(a, b)$, we have*

$$\| L_n f - f \|_p \leqslant A'_p(\lambda_{n,p,0}(\| f \|_1 + \| f' \|_1) + \lambda_{n,1,2}^{\beta_p} \| f' \|_p)$$

with $\beta_p = 1/3$ if $p \leqslant 3$ and $\beta_p = 1/p$ if $p \geqslant 3$.

Theorem 2′. *If, moreover, the sequence (L_n) is uniformly bounded on $L^p(a, b)$, we have for every $f \in L^p(a, b)$:*

$$\| L_n f - f \|_p \leqslant B'_p(\lambda_{n,p,0} \| f \|_1 + \omega_p(f, \lambda_{n,p,0} + \lambda_{n,1,2}^{\beta_p})).$$

We use the same methods as in the proof of Theorems 1 and 1′ with the help of Lemma 1.1 and the following lemma.

Lemma 2.1. *If $f \in W^{1,p}(a, b)$, we have*

$$\| 1_{E_n^c}(x)\, L'_n(f(x) - f(t), x) \|_p \leqslant C'_p \lambda_{n,1,2}^{1/p} \| f' \|_p.$$

with $\beta_p = 1/p$ if $p \geqslant 3$ and $\beta_p = 1/3$ if $p \leqslant 3$.

Proof. With the help of Hölder inequality for the integrals, we write for $f \in W^{1,p}(a, b)$ and $(x, t) \in (a, b)^2$:

$$|f(x)-f(t)|^p \leqslant \|f'\|_p^p \, |x-t|^2 \, (b-a)^{p-3} \qquad \text{if} \quad p \geqslant 3$$

and

$$\left|\int_t^x f'(u) \, 1_{\{u \mid |x-u|>h\}}(u) \, du\right|^p$$
$$\leqslant \|f'\|_p^p \left|\frac{x-t}{h}\right|^{3-p} |x-t|^{p-1} \qquad \text{if} \quad p \leqslant 3.$$

Now, we use Hölder inequality for the linear positive operator L_n. The quantities

$$(b-a)^{3-p} \int_{E_n^c} |L_n^t(f(t)-f(x), x)|^p \, dx \qquad \text{if} \quad p \geqslant 3$$

and

$$\int_{E_n^c} \left| L_n^t \left(\int_t^x f'(u) \, 1_{\{u \mid |x-u|>h\}}(u) \, du, x \right) \right|^p dx \qquad \text{if} \quad p \leqslant 3$$

are bounded by $\|f'\|_p^p \, \|1_{E_n^c} L_n \phi_0\|_\infty^{p-1} \int_a^b L_n^t((x-t)^2, x) \, dx$.

Furthermore, we have

$$\int_a^b L_n^t((x-t)^2, x) \, dx$$
$$= \int_a^b (x^2(L_n\phi_0(x)-\phi_0(x)) - 2x(L_n\phi_1(x)-\phi_1(x)) + L_n\phi_2(x)-\phi_2(x)) \, dx$$
$$\leqslant 4\lambda_{n,1,2}.$$

So, the result is proven if $p \geqslant 3$.

In the case $p \leqslant 3$, we use Lemma 1.2 to bound

$$\|1_{E_n^c}(x) \, L_n^t(f(x)-f(t), x)\|_p \leqslant 4h \, \|f'\|_p + h^{1-3/p} 2^{1+1/p} \lambda_{n,1,2}^{1/p} \, \|f'\|_p.$$

We choose $h = \lambda_{n,1,1}^{1/3}$ and the result follows.

III. Examples

In the following examples, we construct, first, operators for which it is impossible to improve the estimates of Theorems 1, 1′, 2, and 2′. The other examples are the Landau operator and some operators of Bernstein type; the estimates are also pretty good for them.

EXAMPLE 1. On $L^p(-1,1)$, $p \geqslant 1$, we define the sequence of positive linear operators L_n:

$$L_n f(x) = \frac{n}{n-1} \begin{cases} (1+2/n) f(x) + (1/2) \int_{|u|<1/n} f(u)\,du & \text{if } |x| > 1/n \\ (1/2) \int_{|u|>1/n} f(u)\,du & \text{if } |x| < 1/n. \end{cases}$$

These operators are uniformly bounded on $L^p(-1,1)$, $p \geqslant 1$,

$$L_n \phi_0 = \phi_0, \qquad \lambda_{n,1,1} \sim \lambda_{n,1,2} \sim \frac{1}{n}, \qquad \| L_n \phi_2 - \phi_2 \|_p \sim n^{-1/p} \sim \lambda_{n,1,1}^{1/p}.$$

On the spaces $L^p(-1,1)$, $p \geqslant 2$ (respectively $p \geqslant 3$), the order of approximation given by Theorems 1 and 1′ (respectively 2 and 2′) is achieved for the function ϕ_2.

EXAMPLE 2. On $L^p(-1,1)$, $p \geqslant 1$, we define

$$L_n f(x) = \begin{cases} (n a_n/2) \int_{|u|<1/n} f(u)\,du & \text{if } |x| < 1/n \\ f(x) & \text{if } |x| > 1/n, \end{cases}$$

where (a_n) is a sequence of positive real numbers. The operator L_n is linear positive, self-adjoint and

$$\| L_n' \|_p = \sup(a_n, 1), \qquad \| L_n \phi_0 - \phi_0 \|_p \sim \frac{a_n - 1}{n^{1/p}},$$

$$\| L_n \phi_1 - \phi_1 \|_1 \sim \frac{a_n}{n^2}, \qquad \| L_n \phi_2 - \phi_2 \|_1 \sim \frac{a_n}{n^3}.$$

(a) For $a_n = 1$, the sequence L_n preserves the constants. The orders of approximation given by Theorems 1 and 1′ are achieved if $p \leqslant 2$, for the function $f(t) = |t|^{-1/4}$. Indeed $\omega_p(f, t) \sim_{t \to 0} t^{1/p - 1/4}$ and we verify that

$$\| L_n f - f \|_p \geqslant \text{Cte}\, n^{1/4 - 1/p} \sim \omega_p(f, \lambda_{n,1,1}^{1/2}),$$

since $L_n f(x) = (4/3)\, n^{1/4}$, if $|x| < 1/n$, and

$$\| L_n f - f \|_p \geqslant 2^{1/p} \left(\int_0^{1/3^4 n} \left(x^{-1/4} - \frac{4}{3} n^{1/4} \right)^p dx \right)^{1/p} \geqslant \left(\frac{2}{3^4 n} \right)^{1/p} n^{1/4}.$$

(b) For $a_n = n^{1/q}$, $q > 0$, the sequence L_n converges for ϕ_0 in $L^p(-1, 1)$, $p < q$, and for ϕ_1 and ϕ_2 in $L^1(-1, 1)$. So it does converge on $W^{1,p}(-1, 1)$ if $p < q$ (Theorems 1 or 2), but it does not converge on $L^1(-1, 1)$ since it is not uniformly bounded on $L^1(-1, 1)$.

EXAMPLE 3. On $L^p(-1, 1)$, $p \geqslant 1$, we set

$$L_n f(x) = \begin{cases} (n/2) \int_{|u| < 1/n} f(x+u)\, du & \text{if} \quad |x| < 1/n \\ f(x) & \text{if} \quad |x| > 1/n. \end{cases}$$

We have $L_n\phi_0 = \phi_0$, $L_n\phi_1 = \phi_1$, and $\|L_n\|_p \leqslant 2$, $\lambda_{n,1,2} = 2/3n^3$.

For this sequence L_n, the orders of approximation given in Theorems 2 and 2′ are achieved, if $p \leqslant 3$, for the function $f(t) = |t|^{-1/4}$. Indeed, we verify that

$$\|L_n f - f\|_p \geqslant \text{Cte}\, n^{1/4 - 1/p} \sim \omega_p(f, \lambda_{n,1,2}^{1/3}),$$

since

$$L_n f(x) = \frac{2n}{3}\left(\left(\frac{1}{n} + x\right)^{3/4} + \left(\frac{1}{n} - x\right)^{3/4}\right), \qquad \text{if} \quad |x| \leqslant \frac{1}{n},$$

and

$$\|L_n f - f\|_p \geqslant \left(2 \int_0^{1/3^4 n} \left(x^{-1/4} - \frac{2n}{3}\left(\left(\frac{1}{n} + x\right)^{3/4} + \left(\frac{1}{n} - x\right)^{3/4}\right)\right)^p dx\right)^{1/p}$$
$$\geqslant \left(\frac{2}{3^4 n}\right)^{1/p} n^{1/4}.$$

Examples 1 and 3 are adapted from examples of [1, 2].

EXAMPLE 4. The sequence of Bernstein-type operators studied in [3] is defined on $L^1(0, 1)$ by

$$L_n f(x) = (n+1) \sum_{k=0}^{n} p_{nk}(x) \int_0^1 p_{nk}(t) f(t)\, dt,$$

where $p_{nk}(x) = (n!/k!(n-k)!)\, x^k (1-x)^{n-k}$.

We have $L_n\phi_0 = \phi_0$, $\|L_n\phi_1 - \phi_1\|_1 \sim 1/n$. In the case $p \leqslant 2$, we get from Theorem 1′ the inequality:

$$\|L_n f - f\|_p \leqslant \text{Cte}\, \omega_p\left(f, \frac{1}{\sqrt{n}}\right), \qquad \forall f \in L^p(0, 1).$$

It is also true if f is continuous on $[0, 1]$ and $p = \infty$. By interpolation this inequality also holds if $p > 2$.

EXAMPLE 5. Here, we study an example of Korovkin operators defined by even functions: Landau operators. They are defined on $L^1(0, 1)$ by

$$L_n f(x) = \rho_n^{-1} \int_0^1 (1-(t-x)^2)^n f(t)\, dt, \qquad \text{where} \quad \rho_n = \int_{-1}^1 (1-t^2)^n\, dt.$$

They are contractions on $L^p(0, 1)$, $1 \leqslant p \leqslant \infty$, they converge uniformly to identity on $[\delta, 1-\delta]$, $0 < \delta < \frac{1}{2}$, but not on $[0, 1]$. They converge on $L^p(0, 1)$, $p \geqslant 1$ (cf. R. G. Mamedov [7] or B. Wood [9]). We get an estimate of the order of approximation that cannot be improved if $p \geqslant 2$. We verify that

$$\| L_n\phi_0 - \phi_0 \|_1 \sim \| L_n\phi_1 - \phi_1 \|_1 \sim n^{-1/2}.$$

Indeed we have

$$\int_0^1 | L_n\phi_0(x) - \phi_0(x)|\, dx = \rho_n^{-1} \int_0^1 \left(\int_{-1}^{-x} (1-u^2)^n\, du + \int_{1-x}^1 (1-u^2)^n\, du \right) dx$$

$$= 2\rho_n^{-1} \int_0^1 \int_x^1 (1-u^2)^n\, du\, dx = [\rho_n(n+1)]^{-1}$$

and

$$\rho_n = \frac{\Gamma(n+1)\,\Gamma(1/2)}{\Gamma(n+3/2)} \underset{n\to\infty}{\sim} \sqrt{\frac{\pi}{n}}.$$

Since $\| L_n\phi_0 \|_\infty \leqslant 1$, at once we get $\lambda_{n,p,0} \leqslant \text{Cte}\, \lambda_{n,1,0}^{1/p}$. On the other hand

$$\lambda_{n,p,0} \geqslant \rho_n^{-1} \left(\int_0^1 \left(\int_x^1 (1-u^2)^n\, du \right)^p dx \right)^{1/p}$$

$$\geqslant \rho_n^{-1} \left(\int_0^{\rho_n/4} \left(\int_0^1 (1-u^2)^n\, du - \int_0^x (1-u^2)^n\, du \right)^p dx \right)^{1/p}$$

$$\geqslant \frac{1}{4} \left(\frac{\rho_n}{4} \right)^{1/p}.$$

Hence $\lambda_{n,p,0} \sim n^{-1/2p}$.

Now, we write, for every $x \in (0, 1)$:

$$\begin{aligned} &L_n\phi_1(x) - \phi_1(x) \\ &\quad = [2(n+1)\,\rho_n]^{-1}\, [(1-x^2)^{n+1} - (1-(1-x)^2)^{n+1}] \\ &\qquad + x(L_n\phi_0(x) - \phi_0(x)). \end{aligned}$$

It follows that

$$\| L_n\phi_1 - \phi_1 \|_p^p \sim \left(\int_0^1 x^p \, | L_n\phi_0(x) - \phi_0(x) |^p \, dx \right)$$
$$\leqslant \| L_n\phi_0 - \phi_0 \|_p^p,$$

and

$$\geqslant \rho_n^{-p} \int_0^1 x^p \left(\int_{1-x}^1 (1-u^2)^n \, du \right)^p dx$$
$$\geqslant \rho_n^{-p} \int_{1-\rho_{n/4}}^1 x^p \left(\int_{\rho_{n/4}}^1 (1-u^2)^n \, du \right)^p dx \sim \rho_n.$$

So $\lambda_{n,p,1} \sim n^{-1/2p}$.

The estimate of Theorem 1′ is, if $f \in L^p(a, b)$,

$$\| L_n f - f \|_p \leqslant B_p(n^{-1/2p} \| f \|_p + \omega_p(f, n^{-1/4})) \qquad \text{if} \quad p \leqslant 2,$$
$$\leqslant B_p(n^{-1/2p} \| f \|_p + \omega_p(f, n^{-1/2p})) \qquad \text{if} \quad p \geqslant 2.$$

The equality holds in the case $p \geqslant 2$ for $f = \phi_1$.

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